Financial mathematics II

Regular payments during the year

Here again, what we have seen before. Let i be the annual interest rate or $q = 1 + i$ the capitalisation factor. Our regular payments P are m times remitted during the year. Then, at the end of the year, with the interest we have

$$P_1 = P \cdot (m + i \cdot \frac{m \pm 1}{2}) \tag{1}$$

$'+'$ applies to movements ahead and $'-'$ to movements behind.

If we already had an initial capital K_0, we have at the end of the year

$$K_1 = K_0 \cdot q + P_1$$

If $P > 0$ we deposit money regularly and if $P < 0$ we withdraw it regularly.

Annual payments at year-end

For years your mother has been putting P_1 = CHF500.– into a savings account for you at the end of each year, which bears interest at $i = 3\%$ *p.a.*. On 31.12. she will make her 20th deposit. How much has been collected in these 20 years with compound interest?

What we have calculated here, we can also calculate formally. Let P_1 be the respective payment on 31.12., i the annual interest rate, or $q = 1 + i$ the capitalisation factor and n the number of years.

$$K_n = P_1 \cdot \frac{1 - q^n}{1 - q} \tag{2}$$

Exercices

(1) Ida Meierhans saves CHF 10,000 at the end of each year for her single-family home. How much will she have saved over 15 years if the annual interest rate is 4%?

(2) Lubomir Lutschenkov saves for his pension over 22 years at a yearly interest rate of 4.5%. He deposits €6,000 at the end of each year. How large will his pension capital be?

(3) Rosvita Sergi has saved a pension capital of CHF150,000 over 19 years. She made fixed annual payments at the end of each year. The annual interest rate was 5%. What was the amount of these regular payments?

Annual payments at the beginning of the year

What changes if the payments are made at the beginning of the year instead of at the end of the year? Every payment at the beginning of the year P_0 has already carried interest until the end of the year, i.e. $P_1 = P_0 \cdot (1+i)$. If we use this in the previous formula, then we have

$$K_n = P_0 \cdot q \cdot \frac{1-q^n}{1-q} \quad \text{where} \quad q = 1+i.$$

Exercices

(4) Solve the previous exercises if the payments were made at the beginning of the year.

Annual disbursements

If I have money (K_0) on an account, it grows yearly by the interest. We have seen that with an annual interest rate i our capital grows after n years to

$$K_n = K_0 \cdot q^n \quad \text{where} \quad q = 1+i.$$

If I transfer money occasionally from this account to another account with *the same* interest rate, the sum of the two accounts is equal to K_n. The money saved on this account is exactly the money that is missing on the other account.

For example, there is CHF100,000 in a pension account. We draw an annual pension of CHF11,000 on January first of each year. The annual interest rate is $i = 3\%$. How much remains after 10 years?

Exercices

(5) Your grandmother has saved CHF314'600. The money bears interest at 2.8% p.a. Your grandmother receives an advance annual pension of CHF30'000.
a. How much is the credit balance after 4 years?
b. What is the credit balance after 11 years?

(6) You have €100'000 debts, with an annual interest rate of 6.2% p.a..
a. How much do you have to repay annually in arrears so that your debt, including interest, is paid off after 10 years?
b. How much would you have to pay in advance?
c. How much would you have to repay annually to pay off your debt in 20 years? Such annual payments are called *annuities.*

Regular payments during the year over several years

If we make m regular payments P during a year, we have calculated in formula (1) the amount P_1 how much we have at the end of the year. This is the money that we pay annually at the end of the year in formula (2). If we already had an initial capital K_0 we have after n years

$$\boxed{K_n = K_0 \cdot q^n + P \cdot \left(m + i \cdot \frac{m \pm 1}{2}\right) \cdot \frac{1 - q^n}{1 - q}}$$

Exercices

(7) Alauda Satiega puts €100 into her savings account at the end of each month, which bears interest at 2% p.a. How much has she saved in 30 years?

(8) Elisabeth Meierhans has CHF677'412.15 on her savings account at an interest rate of 2.4% p.a.. She withdraws a monthly pension of CHF2,800 on the 1st of each month. What remains on her account after 13 years?

(9) See exercise 6. You want to pay off your debt with monthly payments.
a. How much do you have to transfer on the last day of each month that your debt is repaid after 10 years?
b. How much would you have to pay in advance each month to pay off your debt in 20 years?

(10) A pension is called *perpetual,* if the credit balance does not shrink in spite of pension payments. Then at most the interest can be used for the pension payments. The perpetual pension is maximal, if

$$K_1 = K_0.$$

Mrs. Kliubenschädl has €1'200'000 and wants that her three children will once inherit €400'000 each. The bank pays interest on her credit balance at

3.1% p.a. What is the maximal perpetual pension that she can withdraw in advance each month?

Solutions

(1) CHF200'235.90
(2) €217'820.27
(3) CHF4911.75
(4) 1. CHF208'245.30
    2. €227'622.18
    3. CHF4677.85
(5) a. CHF222704.40
    b. CHF35312.75
(6) a. €13'715.83
    b. €12'915.09
    c. €8'343.26 in advance or €8'860.554 in arrears
(7) €49'127.94
(8) CHF409'891.85
(9) a. €1111.40
    b. €714.39
(10) €3'048,80

HES-SO Valais-Wallis • Rte de la Plaine 2 • CP 80 • 3960 Sierre
+41 58 606 89 11 • info@hevs.ch • www.hevs.ch